MATH5633 Loss Models I Autumn 2024

Chapter 5: Risk Measures

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Preview

This chapter introduces risk measures, which are used to assess potential losses and uncertainty. We will begin by defining the concept of risk measures, including coherent risk measures, and then introduce several popular examples of risk measures.

Key topics in this chapter:

1. Risk measures and coherent risk measures;
2. Value-at-Risk (VaR);
3. Conditional Tail Expectation (CTE)/Expected Shortfall (ES);
4. Tail-Value-at-Risk (TVaR).

1 Risk Measures

A risk measure plays a crucial role in determining capital requirements, pricing, and reserve setting. They are used to quantify the magnitude of risks. Broadly speaking, a risk measure is a mapping from a loss variable X to a real number:

Definition 1.1 A risk measure of a loss variable X is a function $\mathcal{H}$ which maps X to a real number: $X \mapsto \mathcal{H}(X) \in \mathbb{R}$.

Under this definition, the expected value $\mathbb{E}[X]$, the variance $\mathrm{Var}[X]$, and the standard deviations $\sqrt{\mathrm{Var}[X]}$ are all risk measures.

Risk measures are first used to develop premium principles. Let X be a loss variable, below lists some examples of premium principles:

- **The expected value premium principle:**

$$\mathcal{H}(X) = (1+\theta)\mathbb{E}[X],$$

where $\theta \geq 0$ is called the *risk loading*;

- **The standard deviation premium principle:**

$$\mathcal{H}(X) = \mathbb{E}[X] + \alpha\sqrt{\text{Var}[X]},\ \ \alpha \geq 0;$$

- **The variance premium principle:**

$$\mathcal{H}(X) = \mathbb{E}[X] + \alpha\text{Var}[X],\ \ \alpha \geq 0.$$

In these examples, we often refer to the difference of the premium and the expected loss as the *premium loading*:

$$\text{Premium Loading} = \mathcal{H}(X) - \mathbb{E}[X].$$

1.1 Coherent Risk Measures

As we have discussed, risk measures can be broadly defined. But why don't people rely solely on expected value and variance? In order to quantify risks in a meaningful manner, it is expected that a risk measure shall satisfy certain desirable properties. To name a few:

- The more "risky" the loss, the higher the value of $\mathcal{H}(X)$;
- Ability to measure the tail behaviour of the loss;
- Computational convenience;
- Interpretability;
- Ability to capture diversification benefit.

Indeed, there is not a single risk measure that is universally better than others, as risk is a rather subjective concept. However, for quantitative risk management, there are 4 widely agreed axioms that a "desirable" risk measure should fulfill. Such risk measures are called ***coherent risk measures***:

Definition 1.2 A risk measure $\mathcal{H}$ is said to be a ***coherent risk measure*** if it satisfies the following 4 properties:

1. ***Translation Invariance:*** for any constant c,

$$\boxed{\mathcal{H}(X + c) = \mathcal{H}(X) + c.}$$

2. ***Positive Homogeneity:*** for any positive constant $\lambda > 0$,

$$\boxed{\mathcal{H}(\lambda X) = \lambda\mathcal{H}(X).}$$

3. ***Monotonicity:*** for any random variables X, Y with $\mathbb{P}(X \leq Y) = 1$,

$$\boxed{\mathcal{H}(X) \leq \mathcal{H}(Y).}$$

4. ***Subadditivity:*** or any random variables X, Y

$$\boxed{\mathcal{H}(X+Y) \leq \mathcal{H}(X) + \mathcal{H}(Y).}$$

Each property in Definition 1.2 can be interpreted as follows:

- ***Translation Invariance:*** This means that adding a constant amount to a risk adds the same amount to the required capital, since a constant has no variation or randomness.
- ***Positive Homogeneity:*** This implies that changing the units of loss (e.g., currency) does not change the risk measure and the required capital (up to a constant).
- ***Monotonicity:*** The bigger the loss, the higher the reserve we need.
- ***Subadditivity:*** Combining losses can result in diversification and reducing the total risk measure. In other words, diversification (i.e. consolidating risks) cannot make the risk greater, but it might make the risk smaller if one risk can hedge the other risk.

Remark 1.1. Under Definition 1.2, the expected value is indeed a coherent risk measure (show it!). However, the expected value is not sufficient to describe the tail behaviour of a loss, which is one major interest in risk management.

In the upcoming sections, we shall introduce several popular risk measures used in actuarial science and risk management, despite not all of them are coherent risk measures.

2 Value-at-Risk

The ***Value-at-Risk (VaR)*** is a very popular risk measure in the finance and insurance industry. It represents the amount of capital required to ensure, with a high degree of certainty, that the company has the ability to absorb the loss and does not become insolvent. The VaR is widely used to meet regulatory and disclosure requirements, including those under Basel II.

Definition 2.1 Let $\alpha \in (0, 1)$. The ***Value-at-Risk*** of the loss variable X is defined as

$$\boxed{\mathrm{VaR}_\alpha(X) := \inf\{x : F_X(x) \geq \alpha\}.}$$

As introduced in Chapter 1, the α-VaR is equivalent to the 100α-th percentile of X. Let's recall some properties of the VaR:

1. If F_X is continuous and strictly increasing, then the inverse F_X^{-1} exists, and $\mathrm{VaR}_\alpha(X) = F_X^{-1}(\alpha)$.
2. If X is continuous, $\mathbb{P}(X \leq \mathrm{VaR}_\alpha(X)) = \alpha$.
3. In general, $\mathbb{P}(X \leq \mathrm{VaR}_\alpha(X)) \geq \alpha$.

Example 2.1 Let $X \sim \mathrm{Pareto}(\alpha, \theta)$. For $p \in (0, 1)$, find $\mathrm{VaR}_p(X)$.

<u>*Solution:*</u>
For $X \sim \mathrm{Pareto}(\alpha, \theta)$,

$$F_X(x) = 1 - \left(\frac{\theta}{x+\theta}\right)^{\alpha}, \ x > 0.$$

By solving

$$p = F_X(x) = 1 - \left(\frac{\theta}{x+\theta}\right)^{\alpha},$$

we obtain

$$\boxed{\mathrm{VaR}_p(X) = F_X^{-1}(p) = \frac{\theta\left[1-(1-p)^{\frac{1}{\alpha}}\right]}{(1-p)^{\frac{1}{\alpha}}}}.$$

In the following, we show that VaR is NOT a coherent risk measure. Indeed, VaR only satisfies the first 3 properties in Definition 1.2, but not the subadditivity.

Proposition 2.1 The Value-at-Risk is translation invariant, positive homogeneous, and monotonic.

Proof.

1. VaR is translation invariant: for any $c \in \mathbb{R}$,

$$\begin{aligned}
\mathrm{VaR}_\alpha(X+c) &= \inf\{x : \mathbb{P}(X + c \leq x) \geq \alpha\} \\
&= \inf\{x : \mathbb{P}(X \leq x - c) \geq \alpha\} \\
&= \inf\{x - c : \mathbb{P}(X \leq x - c) \geq \alpha\} + c \\
&= \inf\{y : \mathbb{P}(X \leq y) \geq \alpha\} + c \\
&= \mathrm{VaR}_\alpha(X) + c.
\end{aligned}$$

2. VaR is positive homogeneous: for any $\lambda > 0$,

$$\mathrm{VaR}_\alpha(\lambda X) = \inf\{x : \mathbb{P}(\lambda X \leq x) \geq \alpha\}$$

$$
\begin{aligned}
&= \inf\{x : \mathbb{P}(X \le \lambda^{-1}x) \ge \alpha\} \\
&= \lambda \inf\{\lambda^{-1}x : \mathbb{P}(X \le \lambda^{-1}x) \ge \alpha\} \\
&= \lambda \inf\{y : \mathbb{P}(X \le y) \ge \alpha\} \\
&= \lambda \mathrm{VaR}_\alpha(X).
\end{aligned}
$$

3. VaR is monotonic: for any random variables X, Y such that $\mathbb{P}(X \le Y)$, we must have $\{Y \le x\} \subseteq \{X \le x\}$ for any $x \in \mathbb{R}$. Hence, $\mathbb{P}(X \le x) \ge \mathbb{P}(Y \le x)$. This implies that, for any $\alpha \in (0,1)$, $\mathbb{P}(Y \le x) \ge \alpha \Rightarrow \mathbb{P}(X \le x) \ge \alpha$, and thus

$$\{x : \mathbb{P}(Y \le x) \ge \alpha\} \subseteq \{x : \mathbb{P}(X \ge x) \le \alpha\}.$$

Therefore,

$$\mathrm{VaR}_\alpha(Y) = \inf\{x : \mathbb{P}(Y \le x) \ge \alpha\} \ge \inf\{x : \mathbb{P}(X \le x) \ge \alpha\} = \mathrm{VaR}_\alpha(X).$$

□

The next example[1] shows that VaR is not subadditive:

Example 2.2 Let B_i, $i = 1, \dots, 1000$, be i.i.d. random variables with $B_i \sim$ Bernoulli(0.1). Let $X_i := 90B_i - 9$.

(a) Find $\mathrm{VaR}_{0.9}(X_1)$.

(b) Let $\bar{X} := \sum_{i=1}^{1000} X_i/1000$. Find $\mathrm{VaR}_{0.9}(\bar{X})$.

(c) Using (a) and (b), show that VaR is not subadditive.

Solution:

(a) The pmf of X_1 is given by $\mathbb{P}(X_1 = -9) = 0.9$, and $\mathbb{P}(X_1 = 81) = 0.1$. Hence, $\mathrm{VaR}_{0.9}(X_1) = -9$.

(b) Notice that $\sum_{i=1}^{1000} X_i = 90\sum_{i=1}^{1000} B_i - 90000 = 90B - 9000$, where $B \sim$ Bin(1000, 0.1). One can verify, by using computer algorithm, that $\mathrm{VaR}_{0.9}(B) = 112$. Therefore, by the translation invariance and positive homogeneity, we have

$$\mathrm{VaR}_{0.9}(\bar{X}) = \frac{90\mathrm{VaR}_{0.9}(B) - 9000}{1000} = \frac{90(112) - 9000}{1000} = 1.08.$$

(c) Assume the contrary that VaR is subadditive, then we would have

$$\mathrm{VaR}_{0.9}(\bar{X}) = \frac{1}{1000}\mathrm{VaR}_{0.9}\left(\sum_{i=1}^{1000} X_i\right) \le \frac{1}{1000}\sum_{i=1}^{1000}\mathrm{VaR}_{0.9}(X_i) = \mathrm{VaR}_{0.9}(X_1),$$

which implies $1.08 = \mathrm{VaR}_{0.9}(\bar{X}) \le \mathrm{VaR}_{0.9}(X_1) = -9$, which is absurd.

[1] The example is adopted from *An Introduction to Computational Risk Management of Equity-Linked Insurance* (Feng, 2018).

3 Conditional Tail Expectation

Let $\alpha \in (0,1)$. The $\alpha-$***conditional tail expectation (CTE)*** is the expected loss given that the loss is greater than the α-VaR:

Definition 3.1 The ***conditional tail expectation*** (a.k.a. ***expected shortfall***) of the loss X at the confidence level $\alpha \in (0,1)$ is defined as

$$\boxed{\mathrm{CTE}_\alpha(X) := \mathbb{E}[X|X > \mathrm{VaR}_\alpha(X)].}$$

The CTE measures the average loss in the worst $100(1-\alpha)\%$ of cases. Compared with VaR, it is more indicative on the tail behaviour of the loss X, as it captures the severity of losses beyond the VaR threshold. It is also clear from definition that

$$\boxed{\mathrm{CTE}_\alpha(X) = \mathbb{E}[X|X > \mathrm{VaR}_\alpha(X)] > \mathbb{E}[\mathrm{VaR}_\alpha(X)|X > \mathrm{VaR}_\alpha(X)] = \mathrm{VaR}_\alpha(X).}$$

Hence, the CTE is more *conservative* than the VaR at the same confidence level α. By relating with the mean excess loss $e_X(\cdot)$, the CTE can be computed using

$$\boxed{\mathrm{CTE}_\alpha(X) = \mathrm{VaR}_\alpha(X) + e_X(\mathrm{VaR}_\alpha(X)) = \mathrm{VaR}_\alpha(X) + \frac{\mathbb{E}\left[(X - \mathrm{VaR}_\alpha(X))_+\right]}{1 - F_X(\mathrm{VaR}_\alpha(X))}.} \tag{1}$$

Example 3.1 Let $X \sim \mathrm{Pareto}(\alpha, \theta)$, where $\alpha > 1$. For $p \in (0,1)$, compute $\mathrm{CTE}_p(X)$.
<u>*Solution:*</u>
For $X \sim \mathrm{Pareto}(\alpha, \theta)$, we know that $\mathbb{E}[X] = \theta/(\alpha - 1)$. From Example 2.1, we also have

$$\mathrm{VaR}_p(X) = \frac{\theta\left[1 - (1-p)^{\frac{1}{\alpha}}\right]}{(1-p)^{\frac{1}{\alpha}}}.$$

Using the formula of $e_X(d)$ for the Pareto distribution (see Chapter 2, Example 2.2),

$$\begin{aligned}
\mathbb{E}[X - \mathrm{VaR}_p(X)|X > \mathrm{VaR}_p(X)] &= \frac{\theta + \mathrm{VaR}_p(X)}{\alpha - 1} \\
&= \frac{\theta(1-p)^{\frac{1}{\alpha}} + \theta\left[1 - (1-p)^{\frac{1}{\alpha}}\right]}{(\alpha-1)(1-p)^{\frac{1}{\alpha}}} \\
&= \frac{\theta}{(\alpha-1)(1-p)^{\frac{1}{\alpha}}}.
\end{aligned}$$

Hence,

$$\begin{aligned}\mathrm{CTE}_p(X) &= \mathbb{E}[X - \mathrm{VaR}_p(X)|X > \mathrm{VaR}_p(X)] + \mathrm{VaR}_p(X) \\ &= \frac{\theta}{(\alpha-1)(1-p)^{\frac{1}{\alpha}}} + \frac{\theta\left[1-(1-p)^{\frac{1}{\alpha}}\right]}{(1-p)^{\frac{1}{\alpha}}} \\ &= \boxed{\frac{\theta}{(1-p)^{\frac{1}{\alpha}}}\left[\frac{\alpha}{\alpha-1} - (1-p)^{\frac{1}{\alpha}}\right]}.\end{aligned}$$

The CTE is NOT a coherent risk measures. In particular, it is NOT sub-additive.

Proposition 3.1 The CTE is translation invariant and positive homogeneous.

Proof.

1. CTE is translation invariant: for any $c \in \mathbb{R}$,

$$\begin{aligned}\mathrm{CTE}_\alpha(X+c) &= \mathbb{E}[X+c|X+c > \mathrm{VaR}_\alpha(X+c)] \\ &= \mathbb{E}[X|X+c > \mathrm{VaR}_\alpha(X)+c] + c \\ &= \mathbb{E}[X|X > \mathrm{VaR}_\alpha(X)] + c \\ &= \mathrm{CTE}_\alpha(X) + c,\end{aligned}$$

   where the second equality follows from the translation invariance of $\mathrm{VaR}_p(X)$.

2. CTE is positive homogeneous: for any $\lambda > 0$,

$$\begin{aligned}\mathrm{CTE}_\alpha(\lambda X) &= \mathbb{E}[\lambda X|\lambda X > \mathrm{VaR}_\alpha(\lambda X)] \\ &= \lambda\mathbb{E}[X|\lambda X > \lambda\mathrm{VaR}_\alpha(X)] \\ &= \lambda\mathbb{E}[X|X > \mathrm{VaR}_\alpha(X)] \\ &= \lambda\mathrm{CTE}_\alpha(X),\end{aligned}$$

   where the second equality follows from the positive homogeneity of $\mathrm{VaR}_p(X)$.

□

Indeed, one can construct a counterexample to show that CTE is not subadditive, the details are omitted herein.

Proposition 3.2 The CTE is not subadditive.

4 Tail-Value-at-Risk

We have seen that VaR does not offer any information on the tail of the underlying risk. While the CTE addresses this shortcoming, it still lacks the subadditivity. When used as a basis of capital requirement, CTE does not reward risk diversification. A similar quantity, known as the ***Tail-Value-at-Risk***, addresses both shortcomings of the VaR and the CTE.

Definition 4.1 The ***Tail-Value-at-Risk*** of the loss X at the confidence level $\alpha \in (0,1)$ is defined as

$$\boxed{\mathrm{TVaR}_\alpha(X) := \frac{1}{1-\alpha}\int_\alpha^1 \mathrm{VaR}_p(X)dp.}$$

The TVaR highly resembles the CTE, as it also measures the average tail loss of X beyond the confidence level α. Indeed, if X is continuous, CTE and TVaR coincides. We first show the following formula:

Proposition 4.1 For any $\alpha \in (0,1)$, we have

$$\boxed{\mathrm{TVaR}_\alpha(X) = \mathrm{VaR}_\alpha(X) + \frac{1}{1-\alpha}\mathbb{E}\left[(X - \mathrm{VaR}_\alpha(X))_+\right].} \tag{2}$$

Proof. Since $\mathrm{VaR}_p(X)$ is the reflection of $F_X(x)$ along the line $y = x$ (see Figure 1, we can see that

$$\begin{aligned}\int_\alpha^1 \mathrm{VaR}_p(X)dp &= (1-\alpha)\mathrm{VaR}_\alpha(X) + \int_{\mathrm{VaR}_\alpha(X)}^\infty S_X(x)dx \\ &= (1-\alpha)\mathrm{VaR}_\alpha(X) + \mathbb{E}\left[(X - \mathrm{VaR}_\alpha(X))_+\right].\end{aligned}$$

Hence,

$$\begin{aligned}\mathrm{TVaR}_\alpha(X) &= \frac{1}{1-\alpha}\int_\alpha^1 \mathrm{VaR}_p(X)dp \\ &= \mathrm{VaR}_\alpha(X) + \frac{1}{1-\alpha}\mathbb{E}\left[(X - \mathrm{VaR}_\alpha(X))_+\right].\end{aligned}$$

□

Proposition 4.2 Suppose that X is a continuous random variable. Then, for any $\alpha \in (0,1)$, $\mathrm{TVaR}_\alpha(X) = \mathrm{CTE}_\alpha(X)$.

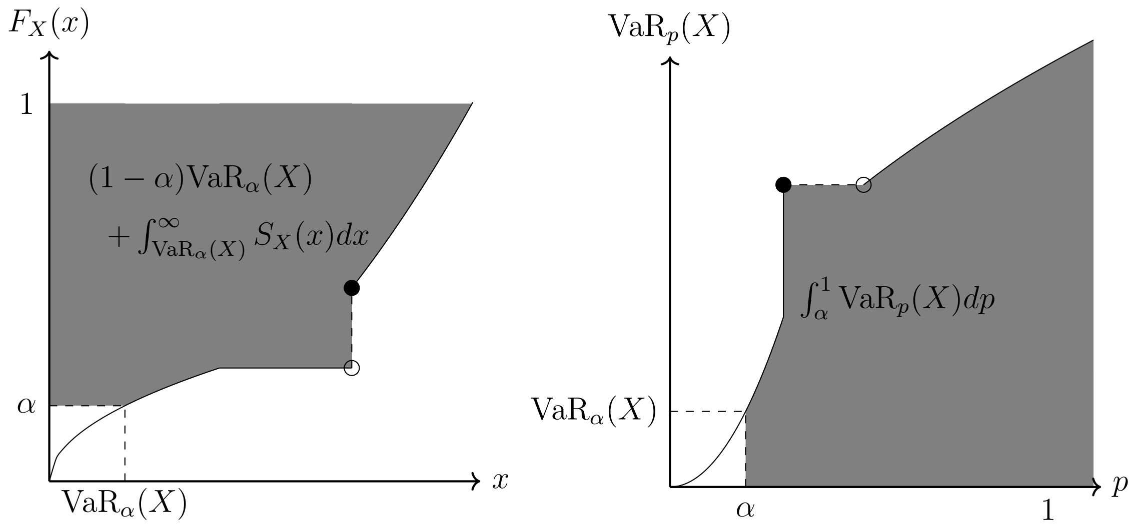

Figure 1: Integrals of Value-at-Risk

Proof. If X is continuous, we have

$$S_X\left(\mathrm{VaR}_\alpha(X)\right) = \mathbb{P}(X > \mathrm{VaR}_\alpha(X)) = 1-\alpha.$$

Using this and Equations 1-(2), we have

$$\begin{aligned}
\mathrm{TVaR}_\alpha(X) &= \mathrm{VaR}_\alpha(X) + \frac{1}{1-\alpha}\mathbb{E}\left[(X-\mathrm{VaR}_\alpha(X))_+\right] \\
&= \mathrm{VaR}_\alpha(X) + \frac{1}{S_X\left(\mathrm{VaR}_\alpha(X)\right)}\mathbb{E}\left[(X-\mathrm{VaR}_\alpha(X))_+\right] \\
&= \mathrm{VaR}_\alpha(X) + e_X\left(\mathrm{VaR}_\alpha(X)\right) \\
&= \mathrm{CTE}_\alpha(X).
\end{aligned}$$

□

Unlike CTE and VaR, TVaR is subadditive. The proof is based on the fact that TVaR is a *distortion risk measure* with a concave distortion function, which is out of the scope of the course. In addition, it is straightforward to show that TVaR is translation invariant, positive homogeneous, and monotonic. Therefore, TVaR is indeed a coherent risk measure.

Theorem 4.3 The TVaR is translation invariant, positive homogeneous, monotonic, and subadditive. Therefore, TVaR is a coherent risk measure.

Example 4.1 Let $X \sim \mathcal{N}(\mu, \sigma^2)$. For any $\alpha \in (0,1)$, let z_α be such that $\Phi(z_\alpha) = \mathbb{P}(Z \le z_\alpha) = \alpha$, i.e., z_α is the α-quantile of the standard normal distribution. Express the following in terms of z_α:

(a) $\mathrm{VaR}_\alpha(X)$;

(b) $\mathrm{TVaR}_\alpha(X)$.

<u>*Solution*</u>:

(a) The cdf of X is given by

$$F_X(x) = \mathbb{P}(X \le x) = \mathbb{P}\left(Z \le \frac{x-\mu}{\sigma}\right) = \Phi\left(\frac{x-\mu}{\sigma}\right).$$

By solving

$$\alpha = F_X(x) = \Phi\left(\frac{x-\mu}{\sigma}\right),$$

we have

$$\frac{x-\mu}{\sigma} = z_\alpha \Rightarrow \boxed{\mathrm{VaR}_\alpha(X) = \mu + \sigma z_\alpha}.$$

(b) We first find $\mathrm{TVaR}_\alpha(Z)$, where $Z \sim \mathcal{N}(0,1)$. Notice that $\mathrm{VaR}_\alpha(Z) = z_\alpha$, and

$$\begin{aligned}
\mathbb{E}\left[(Z - \mathrm{VaR}_\alpha(Z))_+\right] &= \int_{z_\alpha}^{\infty} \frac{x - z_\alpha}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \\
&= \frac{1}{\sqrt{2\pi}} \int_{z_\alpha}^{\infty} z e^{-\frac{z^2}{2}} dz - z_\alpha \int_{z_\alpha}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \\
&= \frac{1}{\sqrt{2\pi}} e^{-\frac{z_\alpha^2}{2}} - z_\alpha \left(1 - \Phi(z_\alpha)\right) \\
&= \phi(z_\alpha) + z_\alpha(\alpha - 1),
\end{aligned}$$

where $\phi(\cdot)$ is the pdf of the standard normal distribution. Using (2), we have

$$\mathrm{TVaR}_\alpha(Z) = z_\alpha + \frac{\phi(z_\alpha) + z_\alpha(\alpha-1)}{1-\alpha} = \frac{\phi(z_\alpha)}{1-\alpha}.$$

Since X is continuous, $\mathrm{CTE}_\alpha(X) = \mathrm{TVaR}_\alpha(X)$, and thus

$$\begin{aligned}
\mathrm{TVaR}_\alpha(X) &= \mathbb{E}[X | X > \mathrm{VaR}_\alpha(X)] = \mathbb{E}\left[\mu + \sigma Z | \mu + \sigma Z > \mu + \sigma z_\alpha\right] \\
&= \mu + \sigma \mathbb{E}[Z | Z > z_\alpha] \\
&= \mu + \sigma \mathrm{TVaR}_\alpha(Z) \\
&= \boxed{\mu + \frac{\sigma \phi(z_\alpha)}{1-\alpha}}.
\end{aligned}$$